AN IMBEDDING OF SPACETIMES

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Abstract. It is shown that any two-dimensional spacetimes with compact Cauchy surfaces can be causally isomorphically imbedded into the two-dimensional Einstein's static universe. Also, it is shown that any two-dimensional globally hyperbolic spacetimes are conformally equivalent to a subset of the two-dimensional Einstein's static universe.

1. Introduction

The group of symmetries plays central roles in theoretical physics as well as many branches of mathematics. In this sense, to find or to analyze the structure of symmetry group is important. In [1], it is shown that any two-dimensional spacetimes with non-compact Cauchy surfaces can be causally isomorphically imbedded into $\mathbb{R}^2_1$, and by use of this, in [2], it is shown that the groups of causal automorphisms on two-dimensional spacetimes with non-compact Cauchy surfaces are subgroups of that of $\mathbb{R}^2_1$. In other words, to imbed a spacetime into a larger space with certain structure preserved is important in analysis of structure groups.

In this paper, we show that any two-dimensional spacetimes with compact Cauchy surfaces can be causally isomorphically imbedded into two-dimensional Einstein's static universe. In conclusion, any two-dimensional globally hyperbolic spacetimes can be causally isomorphically imbedded into two-dimensional Einstein's static universe. We also show that the role of causally isomorphic imbedding can be replaced by conformal diffeomorphism.

2. Preliminaries

In this section, we briefly review and improve the results in [1] and [3].

Let M be a spacetime with a non-compact Cauchy surface Σ. For $p \in J^+(\Sigma)$ and $q \in J^-(\Sigma)$, let $S_p^+ = J^-(p) \cap \Sigma$ and $S_q^- = J^+(q) \cap \Sigma$. Then, the compact and connected subsets S_p^+ and S_q^- uniquely determine the points

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p and q, and thus $C_M^+ = \{S_p^+ \mid p \in J^+(\Sigma)\}$ and $C_M^- = \{S_q^- \mid q \in J^-(\Sigma)\}$ represent M.

If $S_p^+, S_q^+ \in C_M^+$ satisfy $S_p^+ \subset S_q^+$, then we can show that $p \leq q$ and similar result also holds for subsets in C_M^-. Also, it is obvious that $S_p^+ \cap S_q^- \neq \emptyset$ if and only if $q \leq p$. In this way, we can encode causal structures of M into its Cauchy surface Σ.

Let N be a spacetime with a non-compact Cauchy surface Σ' and $f : \Sigma \to \Sigma'$ be a homeomorphism. If f induces a bijection from C_M^+ onto C_N^+, and from C_M^- onto C_N^-, then f can be extended to a unique causal isomorphism $\overline{f} : M \to N$. This is Theorem 5.4 in [3].

If a homeomorphism $f : \Sigma \to \Sigma'$ induces a map, not necessarily bijective, from C_M^+ into C_N^+, and from C_M^- into C_N^-, then f can be uniquely extended to a causally isomorphic imbedding $\overline{f} : M \hookrightarrow N$. In other words, $\overline{f}$ is an imbedding and satisfies $x \leq y$ if and only if $\overline{f}(x) \leq \overline{f}(y)$ for all x and y in M.

Let $\mathbb{R}_1^2 = \{(t, x) \mid t, x \in \mathbb{R}\}$ be a two-dimensional Minkowski spacetime. Then $\mathbb{R}_{t_0} = \{(t_0, x) \mid x \in \mathbb{R}\}$ is a Cauchy surface of $\mathbb{R}_1^2$. One of the characteristic properties of $\mathbb{R}_1^2$ is that for any compact and connected subset A of $\mathbb{R}_{t_0}$, there exist unique p and q such that $S_p^+ = S_q^- = A$. Therefore, for any two-dimensional spacetime M with a non-compact Cauchy surface Σ, any homeomorphism $f : \Sigma \to \mathbb{R}_{t_0}$ can be uniquely extended to a causally isomorphic imbedding $\overline{f} : M \hookrightarrow \mathbb{R}_1^2$ and thus we have the following.

Theorem 2.1. *For any given homeomorphism $f : \Sigma \to \mathbb{R}_{t_0}$, we can extend f to a causally isomorphic imbeeding $\overline{f} : M \hookrightarrow \mathbb{R}_1^2$. In other words, any two-dimensional spacetime with non-compact Cauchy surfaces can be causally isomorphically imbedded in $\mathbb{R}_1^2$.*

Proof. This is Theorem 5.1 in [1]. □

It is well-known that, if $f : M \to N$ is a causal isomorphism with the dimension of M bigger than two, then f becomes a conformal diffeomorphism. ([4], [5], [6].) However, this is not the case when the dimension is two and so the causal isomorphism given in the above theorem is not necessarily a conformal diffeomorphism and not even necessarily smooth. Nevertheless, even when the dimension is two, if we take a C^∞-diffeomorphism $f : \Sigma \to \mathbb{R}_{t_0}$, then the extended map $\overline{f}$ is a conformal diffeomorphism.

Proposition 2.1. *If we take $f : \Sigma \to \mathbb{R}_{t_0}$ to be C^∞ diffeomorphism, then the induced imbedding $\overline{f}$ is a C^∞-conformal diffeomorphism onto its image in $\mathbb{R}_1^2$.*

Proof. For $p \in J^+(\Sigma)$, there exist unique x and y in Σ such that $\partial S_p^+ = \{x, y\}$ and, since x and y are determined by unique null geodesics from p, the dependence of x and y on p is smooth. Since f is C^∞, the dependence

of $f(x)$ and $f(y)$ on p is also smooth, and by the same argument using null geodesics, the dependence of $\overline{f}(p)$ on $f(x)$ and $f(y)$ is smooth. Therefore, $\overline{f}$ is smooth and likewise, $\overline{f}^{-1}$ is also smooth.

To show that $\overline{f}$ is conformal, it suffices to show that $\overline{f}_*(v)$ is null whenever $v \in T_pM$ is a null vector. Let γ be a null geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then, since Σ is non-compact, there exist no null cut points along γ and so we have $\gamma(0) \leq \gamma(t)$ but not $\gamma(0) << \gamma(t)$ for all t. Since $\overline{f}$ is a causal isomorphism, we have $\overline{f}(\gamma(0)) \leq \overline{f}(\gamma(t))$ but not $\overline{f}(\gamma(0)) << \overline{f}(\gamma(t))$. Therefore, any causal curve from $\overline{f}(\gamma(0))$ to $\overline{f}(\gamma(t))$ is a null pregeodesic and thus $\overline{f}_*(v)$ is null. The same argument applied to $\overline{f}^{-1}$ shows that $\overline{f}^{-1}$ is also conformal. $\square$

Therefore, we have the following.

Theorem 2.2. *Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then M is conformally equivalent to a globally hyperbolic open subset of $\mathbb{R}^2_1$ that contains x-axis as a Cauchy surface.*

Proof. Take $t_0 = 0$ in the previous proposition. $\square$

In fact, if sufficient smoothness is assumed, the study of causal structure and the study of conformal structure are essentially the same as the following theorem shows.

Theorem 2.3. *Let $f : M \to N$ be a conformal diffeomorphism. Then f is either a causal isomorphism or an anti-causal isomorphism.*

Proof. Let M and N be spacetimes in which time-orientations are given by nowhere-vanishing timelike vector fields X and Y, respectively. Let $Z = (f^{-1})_*(Y)$. Then, since $g_M(X, Z) = g_N(f_*(X), Y)$, we can see that, if $g_M(X, Z) = 0$, then $F_*(X)$ is a spacelike vector, which contradicts to that f is conformal. Therefore, $g_N(f_*(X), Y)$ is nowhere zero and so, since $f_*(X)$ is timelike, $f_*(X)$ is either everywhere future-directed or everywhere past-directed.

Let $v \in T_pM$ be a future-directed timelike vector. Then, since $g_N(f_*(v), f_*(X)) = g_M(v, X_p) < 0$, $f_*(v)$ determines the same time-orientation as that determined by $f_*(X)$. Therefore, $f_*(v)$ is future-directed if $f_*(X)$ and Y determine the same time-orientation, and otherwise, $f_*(v)$ is past-directed. $\square$

3. Spacetimes with compact Cauchy surfaces

Let (M, g) be a two-dimensional spacetime with compact Cauchy surfaces. Then by Theorem 1 in [7], M is diffeomorphic to $\mathbb{R} \times S^1$ and we can use (t, e^{ix})as a coordinate on M.

Since $\exp : \mathbb{R} \to S^1$ defined by $\exp(x) = e^{ix}$ is a covering map, the map $\pi : \mathbb{R} \times \mathbb{R} \to M = \mathbb{R} \times S^1$ given by $\pi(t, x) = (t, e^{ix})$ is a covering

map. Let $\overline{M}$ be $\mathbb{R} \times \mathbb{R}$ with the pull-back metric $\pi^* g$. Then, $\overline{M}$ is a universal covering space of M in such a way that π is a time-orientation preserving covering map. Then by Theorem 2.1 in [8] or the proof of Theorem 14 in [9], $\overline{M}$ is globally hyperbolic with the non-compact Cauchy surface $\Sigma = \{(0, x) \mid x \in \mathbb{R}\}$.

If we choose homeomorphism $f : \Sigma \to \mathbb{R}_0$ to be $f(x) = x$, where $\mathbb{R}_0 = \{(0, x) \mid x \in \mathbb{R}\}$ is a Cauchy surface of $\mathbb{R}^2_1$, then f induces a causally isomorphic imbedding $\overline{f} : M \hookrightarrow \mathbb{R}^2_1$.

The two-dimensional Einstein's static universe is $E = \mathbb{R} \times S^1$ with the flat metric $-dt^2 + d\theta^2$. Thus the universal covering space of E is $\mathbb{R}^2_1$ with the covering map $\pi_E(t, \theta) = (t, e^{i\theta})$. Therefore, we have the following diagram.

If we identify $\overline{M} = \mathbb{R} \times \mathbb{R}$ with $\overline{f}(\overline{M})$, then, since $f(x) = x$, we have $\pi = \pi_E$ on $\overline{f}(\overline{M})$, and thus M is causally isomorphic to $\pi_E \circ \overline{f}(\overline{M})$, which is an open subset of E. Therefore, we have the following.

Theorem 3.1. *Any two-dimensional spacetimes with compact Cauchy surfaces can be causally isomorphically imbedded into the two-dimensional Einstein's static universe.*

Since the homeomorphism $f : \Sigma \to \mathbb{R}_0$ used in the above theorem is, in fact a diffeomorphism, by Proposition 2.1, $\overline{f}$ is a smooth conformal diffeomorphism and thus, since the covering maps π and π_E are smooth local isometries, we have the following.

Theorem 3.2. *Any two-dimensional spacetimes with compact Cauchy surfaces can be conformally equivalently imbedded into the two-dimensional Einstein's static universe.*

It is a well-known fact that two-dimensional Minkowski spacetime can be causally isomorphically, or conformally equivalently imbedded into the two-dimensional Einstein's static universe in such a way that $\mathbb{R}_0$ is sent to $S^1 - \{\text{a point}\}$ (See sectin 5.1 in [10]). Then, by combining this with Theorem 2.1 or Theorem 2.2, we have the following.

Theorem 3.3. *Any two-dimensional spacetimes with non-compact Cauchy surfaces can be causally isomorphically, or conformally equivalently imbedded into the two-dimensional Eisntein's static universe.*

If we now combine this theorem with Theorem 3.1 and Theorem 3.2, we have the following.

Theorem 3.4. *Any two-dimensional globally hyperbolic spacetimes can be causally isomorphically, or conformally equivalently imbedded into the two-dimensional Einstein's static universe.*

In this theorem, it must be noted that if spacetimes have non-compact Cauchy surfaces,then the Cauchy surface is sent to $S^1 - \{\text{a point}\}$ in E

and, if they have compact Cauchy surfaces, then the Cauchy surface is sent to S^1.

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